Calculus Notes

1. Summation

$$\sum_{i=m}^{n} a_k$$

* Note *
Even though m can be less than 0, for our course, m must be greater than or equal to 0.

i is the index
m is the inital value
n is the ending value
a_k is the function / general term

Properties of summation.

1.
$$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

Proof:

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1+b_1) + (a_2+b_2) + \ldots + (a_n+b_n) \quad \text{By def of } \Sigma$$
$$= (a_1+a_2+\ldots+a_n) + (b_1+b_2\ldots+b_n)$$
$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

QED

2.
$$\sum_{k=1}^{n} c(a_k) = c\sum_{k=1}^{n} a_k$$

Proof:

$$\sum_{k=1}^{n} c(a_k) = ca_1 + ca_2 \ldots + ca_n \quad (\text{By def of } \Sigma)$$
$$= c(a_1+a_2+\ldots+a_n)$$
$$= c\sum_{k=1}^{n} a_k$$

QED

3. $\sum_{k=1}^{n} a_k = \sum_{k=1}^{l-1} a_k + \sum_{k=l}^{n} a_k$

Proof:

$LS = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots a_n$

$RS = \sum_{k=1}^{l-1} a_k = a_1 + a_2 + \ldots a_{l-1}$

$\sum_{k=l}^{n} a_k = a_l + a_{l+1} \ldots + a_n$

$\sum_{k=1}^{l-1} a_k + \sum_{k=l}^{n} a_k = a_1 + a_2 \ldots a_n$
$= LS$

QED

Sum Formulas

1. $\sum_{k=1}^{n} c = cn$, c is a constant

Proof:

$\sum_{k=1}^{n} c = c + c + c \ldots$ (n times)

$= cn$

QED

2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

You can prove these with induction.

2. Riemann's Sum

A Riemann's sum for a function f on the interval $[a,b]$ is a sum of the form

$$\sum_{k=1}^{n} f(x_i^*)\,\Delta x$$

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a + i\Delta x \quad x_i^* \in [x_{i-1}, x_i]$

Left Riemann Sum

$$\sum_{k=1}^{n} f(x_{k-1})\,\Delta x$$

Mid-Point Riemann Sum

$$\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x$$

Right Riemann Sum

$$\sum_{k=1}^{n} f(x_k)\,\Delta x$$

Upper and Lower Sums

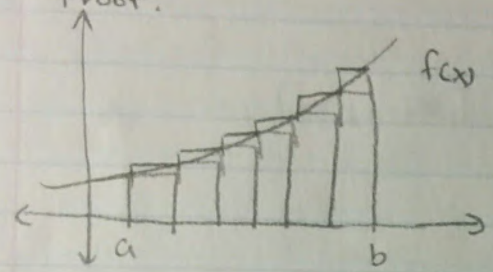
$$U(f,P) = \sum_{i=1}^{n} M_i\,(x_i - x_{i-1})$$

$$L(f,P) = \sum_{i=1}^{n} m_i\,(x_i - x_{i-1})$$

M_i is the sup of each partition.
m_i is the inf of each partition.

Using left Riemann's Sum and right Riemann's sum and the def of increasing to prove that if f is positive and increasing on $[a,b]$ prove that left sum $\leq$ right sum.

Proof:



Let $P = \{x_i\}_{i=1}^{n}$ be any partition of $[a,b]$.

Left Riemann Sum $= \sum_{k=1}^{n} f(x_{i-1})\Delta x$

Right Riemann Sum $= \sum_{k=1}^{n} (f(x_i))\Delta x$

Since $x_{i-1} \leq x_i$, $f(x_{i-1}) \leq f(x_i)$, $\Delta x\, f(x_{i-1}) \leq \Delta x\, f(x_i)$,

$\sum_{i=1}^{n} f(x_{i-1})\Delta x \leq \sum_{i=1}^{n} f(x_i)\Delta x$

$\therefore$ Left Riemann's Sum $\leq$ Right Riemann's Sum

Definite Integrals

Let f be a function defined on $[a,b]$. The definite integral of f from $x=a$ to $x=b$ is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

Properties of Definite Integral

Let $a,b \in \mathbb{R}$ s.t. $a<b$

Suppose f and g are integrable on $[a,b]$, then

1. If $f(x) \geq 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$
2. If $f(x) \leq 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \leq 0$
3. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$, k is a constant
5. $\int_a^a f(x)\,dx = 0$
6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $\forall c \in [a,b]$
7. If $\exists M, N \in \mathbb{R}^+$ s.t. $m \leq f(x) \leq n$, $\forall x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq n(b-a)$
8. If $f(x) \leq g(x)$ $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Definite Integral Formulas

1. $\int_a^b c\,dx = c(b-a)$, $c \in \mathbb{R}$ and c is a constant
2. $\int_a^b x\,dx = \frac{1}{2}(b^2-a^2)$
3. $\int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3)$

Darboux Definition

If $f(x)$ is cont on $[a,b]$ or if $f(x)$ has a finite number of jump discontunities, then f is integrable on $[a,b]$.

Consider $f(x) \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$ (Dirchlet Function)

Darboux's Def states that $f(x)$ is integrable on $[a,b]$ iff $\sup\{L(f,P) \mid \forall P \text{ of } [a,b]\} = \inf\{U(f,P) \mid \forall P \text{ of } [a,b]\}$

Integrability Reformation/ Darboux Integration states that f is integrable on $[a,b]$ iff $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$. s.t. $U(f,P) - L(f,P) < \varepsilon$.

FTOC Part 1

Let $a,b \in \mathbb{R}$ and $a<b$

If f is cont on $[a,b]$ and F is any antiderivate/integral of f on $[a,b]$, then $\int_a^b f(x)\,dx = f(b) - f(a)$.

Proof

Suppose f is cont on $[a,b]$ and F is an integral of f on $[a,b]$.

Let $P = \{x_i\}_{i=0}^{n}$ be any partition of $[a,b]$.

Let $x_i = a + i\Delta x$

Let $\Delta x = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \quad x_i^* \in [x_{i-1}, x_i]$$

F is diff on $[a,b]$ and cont on $[a,b]$ (diff $\rightarrow$ cont).

F is diff and cont on each $[x_{i-1}, x_i] \in [a,b]$.

$\therefore$ MVT is applied to F on each partition.

By MVT, $\exists c \in [x_{i-1}, x_i]$ s.t. $F(c) = \dfrac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$

$$F(c)\underbrace{(x_i - x_{i-1})}_{\Delta x} = F(x_i) - F(x_{i-1})$$

$$\sum_{i=1}^{n} f(c)\Delta x = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))$$

Let $x_i = c$

$$\lim_{n \to \infty} \sum_{i=1}^{n} F(x_i)\Delta x$$

$= (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) \ldots + (F(x_n) - F(x_{n-1}))$ (Telescoping sum)

$= \underset{b}{\underset{\uparrow}{F(x_n)}} - \underset{a}{\underset{\uparrow}{F(x_0)}}$

$= F(b) - F(a)$

QED

MVT for Integrals

Let $a, b \in \mathbb{R}$ s.t. $a < b$

If $f(x)$ is cont on $[a,b]$, then $\int_a^b f(x)\,dx = f(c)(b-a)$ for some $c \in [a,b]$.

Proof

Since f is cont on $[a,b]$, then if $\exists m, n \in \mathbb{R}^+$ s.t. $m \leq f(x) \leq n$, $\forall x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq n(b-a)$.

$$m(b-a) \leq \int_a^b f(x)\,dx \leq n(b-a)$$

$$m \leq \underbrace{\frac{\int_a^b f(x)\,dx}{b-a}}_{\text{Let this be } F} \leq n$$

By the Intermediate Value Theorem, we know $\exists c$ s.t. $m \leq c \leq n$.

$\therefore f(c) = F$

$$= \frac{\int_a^b f(x)\,dx}{b-a}$$

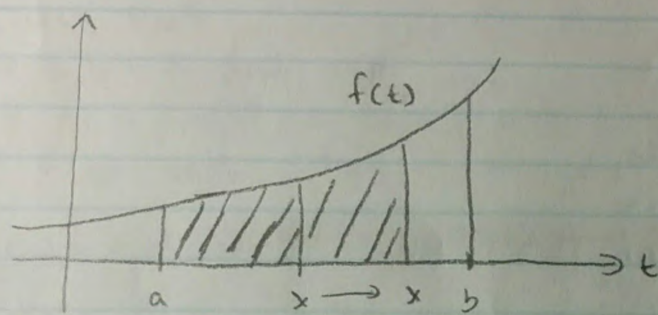
$f(c)(b-a) = \int_a^b f(x)\,dx \Longleftrightarrow \underbrace{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{Used to prove FTOC 2}}$

QED

Area Accumlation Function

Suppose f is cont on $[a,b]$. Then, the area acculmation function for f on $[a,b]$ is the function that, for $x \in [a,b]$, is equal to the signed area between the graph of f and the x-axis on $[a,x]$:

$$A(x) = \int_a^x f(t)\,dt$$



As x increases, the integral of $f(t)$ increases.
As x decreases, the integral of $f(t)$ decreases.

FTOC 2

If f is cont on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt \ \forall x \in [a,b]$, then

a) F is cont and diff on $[a,b]$.

b) $F'(x) = f(x)$

Proof:

Suppose f is cont on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$

Case 1: $x \in (a,b)$

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

If $h>0$,

$$\lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

If $h<0$,

$$\lim_{h\to 0} \frac{-\int_{x+h}^{x} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

If $h>0$

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$
$$= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt$$
$$= \int_x^{x+h} f(t)\,dt$$

If $h<0$

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$
$$= -\int_{x+h}^a f(t)\,dt - \int_a^x f(t)\,dt$$
$$= -\int_{x+h}^x f(t)\,dt$$
$$= \int_x^{x+h} f(t)\,dt$$

$$\therefore \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right) \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} \left(\frac{1}{(x+h)-x}\right) \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c) \quad \text{(By MVT for Integrals)} \quad x \le c \le x+h$$

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c)$$

$$= f(x)$$

Case 2: $x=a$ and $x=b$

An analogus argument shows that the results are true; replace the 2-sided limit with 1-sided limit.

QED

Techniques of Integration

1. Inspection

If the integral is a basic integral, we can solve it by inspection.

E.g. $\int \cos x \, dx = \sin x + C$

2. Substitution

If f and g' are cont on $[a,b]$, then $\int_a^b f(x)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

Proof:

Suppose f and g' are cont on $[a,b]$.
Let F be an integral of f on $[a,b]$.

$$LS = \int_a^b f(x)g'(x)\,dx$$
$$= F(g(x)) \Big|_a^b$$
$$= F(g(b)) - F(g(a))$$

$$RS = \int_{g(a)}^{g(b)} f(u)\,du$$
$$= F(u) \Big|_{g(a)}^{g(b)}$$
$$= F(g(b)) - F(g(a))$$

$LS = RS$

QED

3. Parts

If $u = f(x)$ and $v = g(x)$ are diff, then

$\int u\,dx = uv - \int v\,du$

$\int_a^b u\,dx = uv\Big|_a^b - \int_a^b v\,du$

Proof

Suppose $u = f(x)$ and $v = g(x)$ are diff

By product rule, $f'(x)g(x) + f(x)g'(x) = (fg)'$

$u'v + uv' = (uv)'$

$uv' = (uv)' - u'v$

$= (uv)' - vu'$

$\int uv' = \int (uv)' - \int vu'$

$$\int u\,dv = \int (uv)' - \int v\,du$$
$$= uv - \int v\,du$$

QED

Hints:

1. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$
3. Since a definite integral is almost always a real number, the derivative of a definite integral $= 0$, if $f(x)$ is cont.
4. For FTOC 1, we can assume that the function is cont. But, for FTOC 2 and other cases, we have to prove that the function is cont on $[a,b]$.

Improper Integrals

Type 1

When either the lower bound or the upper bound or both is infinity or negative infinity.

I.e.

$\int_a^\infty f(x)\,dx$, $\int_{-\infty}^a f(x)\,dx$, $\int_{-\infty}^\infty f(x)\,dx$

Case 1

When only 1 of the bounds is infinity or negative infinity

$\int_a^\infty f(x)\,dx$, $\int_{-\infty}^a f(x)\,dx$

Rewrite the integral like this:

$\lim_{A \to \infty} \int_a^A f(x)\,dx$, $\lim_{A \to -\infty} \int_A^a f(x)\,dx$

Then, solve the integral.

Finally, calculate the limit of your answer. If your final answer is a real number, then the integral converges and has a solution. If the final answer is positive or negative infinity, then the limit diverges.

Case 2

If both the upper and lower bounds are infinity or negative infinity.

$\int_{-\infty}^\infty f(x)\,dx$

Choose a number between ∞ and $-\infty$, preferably 0 if you can, and split the integral into 2 integrals.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{c}^{\infty} f(x)\,dx + \int_{-\infty}^{c} f(x)\,dx \quad , \quad c \in (-\infty, \infty)$$

Then, evaluate each new integral like in Case 1. Finally, if either small integral diverges, then the entire thing diverges. If both small integrals converges, then the entire thing converges to their sum.

Type 2

When the integral has a V.A. within the interval of the lower bound to the upper bound, inclusive.

$$\int_0^3 \frac{1}{x}\,dx \quad , \quad \int_0^4 \frac{1}{x-3}\,dx \quad , \quad \int_0^4 \frac{1}{x-4}\,dx$$

$\frac{1}{x}$ has a V.A. at $x=0$, which is the lower bound.

$\frac{1}{x-3}$ has a V.A. at $x=3$, which is in the interval of $[0,4]$.

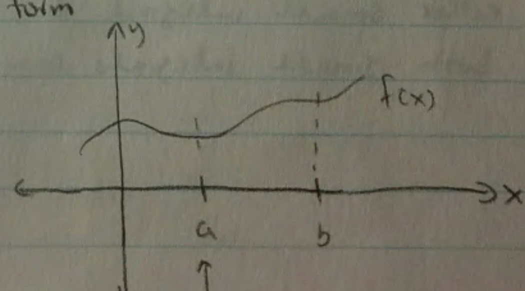
$\frac{1}{x-4}$ has a V.A. at $x=4$, which is the upper bound.

Case 1.

The V.A. is at the lower bound

Rewrite the integral in this form

$$\lim_{A \to a^+} \int_A^b f(x)\,dx$$



Solve the integral and then find the limit of your answer. If the final answer is a real number, the integral converges. If your final answer is $\pm\infty$, the integral div.

We find $\lim_{A \to a^+}$ because the integral is to the right of a.

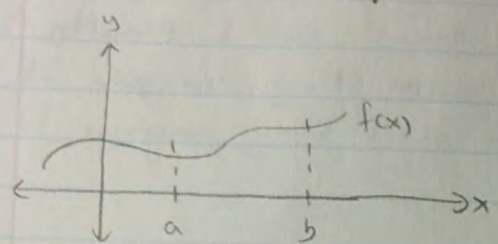
Case 2

The V.A. is at the upper bound.

This is similar to case1, but you do

$$\lim_{A \to b^-} \int_a^A f(x)\,dx$$ instead.

The rest of the steps are the same as case 1.



We find $\lim_{A \to b^-}$ because the integral is on the left of b.

Case 3

The V.A. is in (a,b).

Suppose the V.A. is c, and $c \in (a,b)$.

Split the integral up as such:

$$\int_a^b f(x)\,dx = \int_c^b f(x)dx + \int_a^c f(x)dx$$

Solve $\int_c^b f(x)\,dx$ using case 1.

Solve $\int_a^c f(x)\,dx$ using case 2.

If either small integral div, the whole thing div.

If both small integrals conv, the whole thing conv.

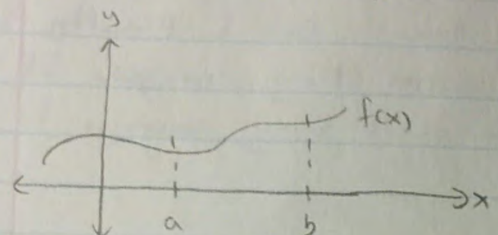
Case 2

The V.A. is at the upper bound.

This is similar to case1, but you do

$\lim_{A \to b^-} \int_a^A f(x)\,dx$ instead.

The rest of the steps are the same as case 1.



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Case 3

The V.A. is in (a,b).

Suppose the V.A. is c, and $c \in (a,b)$.

Split the integral up as such:

$$\int_a^b f(x)\,dx = \int_c^b f(x)dx + \int_a^c f(x)dx$$

Solve $\int_c^b f(x)\,dx$ using case 1.

Solve $\int_a^c f(x)\,dx$ using case 2.

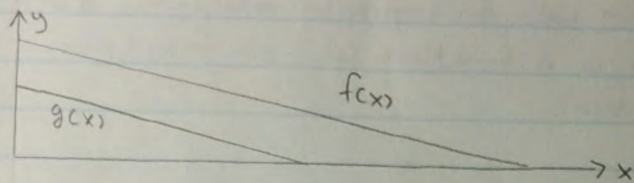
If either small integral div, the whole thing div.

If both small integrals conv, the whole thing conv.

Comparison Theorem

Suppose that f(x) and g(x) are functions that are cont on an interval, I.

1. If the improper integral of f on I converges and $0 \le g(x) \le f(x)$ $\forall x \in I$, then g(x) also converges on I.



Since the area underneath f(x) is finite, the area of g(x) must also be finite.

Proof:

Suppose f(x) is a function with an improper integral, $\int_a^\infty f(x)\,dx$, and it converges. If g(x) is a function s.t. $0 \le g(x) \le f(x)$ $\forall x \in [a, \infty)$ and $B > a$, then:

$$0 \le \int_a^B g(x)\,dx \le \int_a^B f(x)\,dx$$

Let's take the limit as $B \to \infty$

$$0 \le \lim_{B \to \infty} \int_a^B g(x)\,dx \le \lim_{B \to \infty} \int_a^B f(x)\,dx$$

By def

$$0 \le \int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx$$

Since $\int_a^\infty f(x)\,dx$ conv to some number L, then

$\int_a^\infty g(x)\,dx$ must conv to a real number between 0 and L.

QED

2. If the improper integral of $f(x)$ on I div and $0 \leq f(x) \leq g(x)$ $\forall x \in I$, then the improper integral of g on I also div.

Proof:

Suppose $f(x)$ is a function with an improper integral, $\int_a^\infty f(x)\,dx$, and it diverges. If $g(x)$ is a function s.t. $0 \leq f(x) \leq g(x)$ $\forall x \in [a, \infty)$ and $B > a$, then:

$$0 \leq \int_a^B f(x)\,dx \leq \int_a^B g(x)\,dx$$

Let's take the limit as $B \to \infty$

$$0 \leq \lim_{B \to \infty} \int_a^B f(x)\,dx \leq \lim_{B \to \infty} \int_a^B g(x)\,dx$$

By def

$$0 \leq \int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

Since $\int_a^\infty f(x)\,dx$ div, then because $g(x) \geq f(x)$ $\forall x \in I$, $\int_a^\infty g(x)\,dx$ must div, too.

QED

Improper integrals of Power Functions on $[1, \infty)$

1. If $0 < p \leq 1$, then $\frac{1}{x^p} \geq \frac{1}{x}$ $\forall x \in [1, \infty)$ and $\int_1^\infty \frac{1}{x^p}\,dx$ div

2. If $p > 1$, then $\frac{1}{x^p} < \frac{1}{x}$ $\forall x \in [1, \infty)$ and $\int_1^\infty \frac{1}{x^p}\,dx$ conv to $\frac{1}{p-1}$.

Proof of 1

$$\int_1^\infty \frac{1}{x}\,dx$$

$$= \lim_{A\to\infty} \int_1^A \frac{1}{x}\,dx$$

$$= \lim_{A\to\infty} \ln|x| \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\ln|A| - \ln|1|\right)$$

$$= \lim_{A\to\infty} \ln|A|$$

$$= \infty$$

$\therefore \int_1^\infty \frac{1}{x}\,dx$ div

By C.T., If $f(x)$ div and $0 \le f(x) \le g(x)$, then $g(x)$ also div.
If $0 < p < 1$, then $\frac{1}{x^p} > \frac{1}{x}$. $\therefore \int_1^\infty \frac{1}{x^p}\,dx$ div if $0 < p \le 1$.

QED

Proof of 2

If $p > 1$, then

$$\int_1^\infty \frac{1}{x^p}\,dx$$

$$= \lim_{B\to\infty} \int_1^B x^{-p}\,dx$$

$$= \lim_{B\to\infty} \left[\frac{1}{1-p} x^{-p+1}\right]_1^B$$

$$= \lim_{B\to\infty} \left[\frac{1}{1-p} B^{1-p} - \frac{1}{1-p}\right]$$

$$= \frac{1}{p-1}$$ QED

Improper Integrals of Power Functions on [0,1]

1. If $0<p<1$, then $\frac{1}{x^p} < \frac{1}{x}$ $\forall x \in [0,1]$ and $\int_0^1 \frac{1}{x^p}\,dx$ conv to $\frac{1}{1-p}$.
2. If $p \geq 1$, then $\frac{1}{x^p} \geq \frac{1}{x}$ $\forall x \in [0,1]$ and $\int_0^1 \frac{1}{x^p}\,dx$ div.

Proof of 1

If $0<p<1$, then

$$\int_0^1 \frac{1}{x^p}\,dx$$

$$= \lim_{A\to 0^+} \int_A^1 \frac{1}{x^p}\,dx$$

$$= \lim_{A\to 0^+} \left[\frac{x^{-p+1}}{1-p} \right]_A^1$$

$$= \lim_{A\to 0^+} \left[\frac{1}{1-p} \left(\underbrace{1^{-p+1}}_{\downarrow\, 1} - \underbrace{A^{-p+1}}_{\downarrow\, 0} \right) \right]$$

$$= \frac{1}{1-p}$$

QED

Proof of 2

$$\int_0^1 \frac{1}{x}\,dx$$

$$= \lim_{A\to 0^+} \int_A^1 \frac{1}{x}\,dx$$

$$= \lim_{A\to 0^+} \ln|x| \Big|_A^1$$

$$= \lim_{A\to 0^+} \ln|1| - \ln|A|$$

$$= \lim_{A\to 0^+} -\ln|A|$$

$$= \infty$$

$\therefore \int_0^1 \frac{1}{x}\,dx$ div

By C.T., if $f(x)$ div and $0 \leq f(x) \leq g(x)$ then $g(x)$ also div.

If $p>1$, then $\frac{1}{x^p} > \frac{1}{x}$.

$\therefore$ If $p \geq 1$, $\frac{1}{x^p}$ div.

Partial Fraction Integrals

Type 1.
The multiplicity of all the factors of the denominator are 1.

E.g. $\int \frac{x-5}{(x+2)(x-1)}\,dx$, $\int \frac{x+2}{(x^2-3)(x+6)}\,dx$, $\int \frac{x}{(x^3-2)(x-6)}\,dx$

The integrals would be in this form:

$$\int \frac{h(x)}{(f(x))^1 (g(x))^1 \ldots (m(x))^1}\,dx$$

where $f(x)$, $g(x)$ and all the other factors of the denominator have multiplicity 1 and can't be equal to 0.

How to Solve it:
Take $\int \frac{x+5}{(x+2)(x-1)}\,dx$ as an example.

Step 1.
Split the fraction up such that each factor becomes a denominator and write each numerator in a general term such that the degree of the numerator is 1 less than the degree of its denominator.

$$\frac{A}{x+2} + \frac{B}{x-1}$$

Since $x+2$ has a degree of 1, then its numerator should have a degree of 0, or is a constant.

Since $x-1$ also has a degree of 1, its numerator will be a constant.

Note: Do not use a variable more than once. If you used A, then you can't use it again. Also, if the denominator isn't factored, you must factor it first.

$\frac{A}{x-1}$, $\frac{Ax+B}{x^2+3}$, $\frac{Ax^2+Bx+C}{x^3+4}$, $\frac{Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots Z}{x^n + \text{a constant}}$, Z is a constant

If the denominator is linear, then its numerator is a constant.
If the denominator is quadratic, then its numerator is linear.
If the denominator is cubic, then its numerator is quadratic.
If the denominator is quartic, then its numerator is cubic.
And so on.

Step 2.
Add the fractions up. Then, expand the numerator and group like terms.

$$\frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$= \frac{Ax-A+Bx+2B}{(x+2)(x-1)}$$ ← Expanded the numerator

$$= \frac{(A+B)x-A+2B}{(x+2)(x-1)}$$ ← Since I have $Ax+Bx$, I can group them to be $(A+B)x$.

Step 3.
Compare your fraction with the original fraction. Then, create a system of equations and solve for A, B, . . .

$$\frac{(A+B)x-A+2B}{(x+2)(x-1)} = \frac{x+5}{(x+2)(x-1)}$$ ← The denominators are the same, so we can ignore them.

$(A+B)x - A+2B = x+5$ ← Here, since the coefficient of x is 1, $A+B=1$. Furthermore, because the constant is 5, $2B-A=5$.

The 2 equations, Solve for A and B now.

$$\begin{cases} A+B=1 \\ 2B-A=5 \end{cases}$$

$A = 1-B$
$2B-(1-B)=5$
$2B-1+B=5$

$3B=6$
$B=2$
$A=-1$

Step 4.
Now that we have A and B, plug them into their fraction and integrate each fraction.

$$\int \frac{x+5}{(x+2)(x-1)}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x-1}\,dx$$

$$= \int \frac{-1}{x+2}\,dx + \int \frac{2}{x-1}\,dx$$

$$= -\ln|x+2| + 2\ln|x-1| + C$$

Type 2.
One or more of the factors in the denominator has a multiplicity greater than 1.

E.g. $\int \frac{x-5}{(x-2)^2(x+1)}\,dx$, $\int \frac{x+6}{(x^2-3)^3(x^2-6)^2}\,dx$

$(x-2)$ has a multiplicity of 2

(x^2-3) has a multiplicity of 3.

(x^2-6) has a multiplicity of 2.

How to Solve it:
Take $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx$ as an example.

Step 1.
Since the denominator isn't factored, we must factor it first.

$$\int \frac{5x^3-3x^2+2x-1}{(x)^2(x^2+1)}\,dx$$

Now, we can split up each factor into its own fraction.

However, if there is a factor with a multiplicity greater than 1, we have to write each of its terms, starting from multiplicity of 1 to the multiplicity of the factor.

In our case, $(x)^2$ has a multiplicity of 2.
Therefore, our fraction will look like: $\frac{A}{(x)} + \frac{B}{(x)^2} + \frac{Cx+D}{x^2+1}$

Note: Do not confuse multiplicity with the degree of the denominator. If you have $(x)^2$, then x has a degree of 1 and a multiplicity of 2. Therefore, its numerator is a constant. If you have $(x^2+1)^4$, the degree of the function is 2, but its multiplicity is 4. The degree is the highest power INSIDE the brackets. The multiplicity is the exponent OUTSIDE the brackets.

Since I had a factor, x, that has a multiplicity greater than 1, I listed the terms starting from a multiplicity of 1 to the multiplicity of the factor, 2.

E.g. $$\frac{5x}{(x-3)^3(x+6)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{(x+6)}$$

Has a multiplicity of 3.

Starts from a multiplicity of 1, $(x-3)$, then goes to a multiplicity of 2, $(x-3)^2$, and finally ends at a multiplicity of 3, the multiplicity of the factor.

E.g. $$\frac{5x}{(2x+1)^5(x-3)} = \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3} + \frac{D}{(2x+1)^4} + \frac{E}{(2x+1)^5} + \frac{F}{(x-3)}$$

In this case, I wrote the factor starting from multiplicity of 1 and ended at multiplicity of 5.

Note: When you are listing the factors from multiplicity of 1 to its end, each of those terms must have the same type of numerator.
I.e. If the numerator of the 1st term is a constant, then the rest of those terms will also have a numerator with a constant.

Step 2.
Add the fractions, expand the numerator and group like terms.

$$\frac{A}{(x)} + \frac{B}{(x)^2} + \frac{Cx+D}{x^2+1} = \frac{A(x)(x^2+1) + B(x^2+1) + (Cx+D)(x^2)}{(x^2)(x^2+1)}$$

$$= \frac{A(x^3+x) + Bx^2 + B + Cx^3 + Dx^2}{(x^2)(x^2+1)}$$

$$= \frac{(A+C)x^3 + (B+D)x^2 + Ax + B}{(x^2)(x^2+1)}$$

Step 3.
Compare the new fraction with the original fraction, and solve for A, B, C, D.

$(A+C)x^3 + (B+D)x^2 + Ax + B = 5x^3 - 3x^2 + 2x - 1$

$A+C = 5 \longrightarrow C = 3$

$B+D = -3 \longrightarrow D = -2$

$A = 2$

$B = -1$

Step 4.
Plug the values for A, B, C, D back into their fraction and integrate.

$$\int \frac{2}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{3x-2}{x^2+1}\,dx$$

$$= 2\ln|x| - x^{-1} + \int \frac{3x}{x^2+1}\,dx + \int \frac{-2}{x^2+1}\,dx$$

$$= 2\ln|x| - x^{-1} + \frac{3}{2}\ln|x^2+1| - 2\arctan(x) + C$$

Hilary

Type 3.

So far, all our examples have been $f(x) = \frac{P(x)}{Q(x)}$, s.t. the degree of $P(x) <$ the degree of $Q(x)$. These are proper rational functions. However, if the degree of $P(x) \geq$ the degree of $Q(x)$, we have to do long division first.

Step 1.

Do long division.

Take $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx$ as an example.

$$
\begin{array}{r}
x + 1 \\
x^3 - x^2 - x + 1 \overline{\smash{\big)}\, x^4 + 0x^3 - 2x^2 + 4x + 1} \\
-(x^4 - x^3 - x^2 + x) \\
\hline
x^3 - x^2 + 3x + 1 \\
-(x^3 - x^2 - x + 1) \\
\hline
4x
\end{array}
$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx = \int \frac{(x^3 - x^2 - x + 1)(x+1) + 4x}{x^3 - x^2 - x + 1} \, dx$$

$$= \int (x+1) + \frac{4x}{x^3 - x^2 - x + 1} \, dx$$

Now, solve the integral using type 1 or type 2 or other methods.

Note: If there is a term missing from the dividend, the polynominal in the $\overline{\smash{\big)}}$, the we must add a 0(term) to it. E.g. $x^4 - x^2 + 4x + 1$ is missing an x^3 term, so I added $0x^3$ to it.

Furthermore, Dividend = (Divisor)(Quotient) + Remainder.

E.g. $\underbrace{x^4 - 2x^2 + 4x + 1}_{\text{Dividend}} = \underbrace{(x^3 - x^2 - x + 1)}_{\text{Divisor}} \underbrace{(x+1)}_{\text{Quotient}} + \underbrace{4x}_{\text{Remainder}}$

Trig Integrals

Useful Trig Indentities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $\cot^2 x + 1 = \csc^2 x$
4. $\cos^2 x = \frac{1+\cos(2x)}{2}$
5. $\sin^2 x = \frac{1-\cos(2x)}{2}$
6. $\sin(2x) = 2\sin x \cos x$

Type 1
Odd/Even Case

Type 2
odd/odd Case

Type 3
Even/Even Case

Trig Substitution

Use it when you see any of these forms:
Let $a \in \mathbb{R}$, $u =$ a polynominal expression with a variable

1. $a^2 + u^2$ Constant squared + variable expression squared
   Let $u = a\tan\theta$

2. $a^2 - u^2$ Constant squared − variable expression squared
   Let $u = a\sin\theta$

3. $u^2 - a^2$ Variable expression squared − Constant squared
   Let $u = a\sec\theta$

E.g.

1. $$\int \frac{3x+1}{\sqrt{x^2+9}}\,dx$$

Because x^2+9 is of the form a^2+u^2, with $a=3$ and $u=x$, let $x=3\tan\theta$.

2. $$\int \frac{(x+3)(4x^2-16)^{5/3}}{\sqrt{x}}\,dx$$

Because $4x^2-16=(2x)^2-4^2$, this is of the form u^2-a^2, with $a=4$ and $u=2x$, let $2x=4\sec\theta$

3. $$\int \frac{1}{\sqrt{2-(3x+1)^2}}\,dx$$

$2-(3x+1)^2=(\sqrt{2})^2-(3x+1)^2$. This is of the form a^2-u^2, with $a=\sqrt{2}$ and $u=3x+1$. Let $3x+1=\sqrt{2}\sin\theta$

Note: Trig sub is the longest way to solve an integer. If there's a faster way, do that.

E.g.

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx$$

This is of the form a^2+u^2, with $a=1$ and $u=x$.

Let $x=a\tan\theta$

$=\tan\theta$

$dx=\sec^2\theta\,d\theta$

Note: $x=\tan\theta$, so $x^2=\tan^2\theta$ and $x^2+1=\tan^2\theta+1=\sec^2\theta$

$$\int \frac{\tan^2\theta}{\sqrt{\sec^2\theta}} \sec^2\theta \, d\theta$$

$$= \int \frac{\tan^2\theta}{|\sec\theta|} \sec^2\theta \, d\theta$$

$\sec\theta > 0$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

If you write this ↑, you can assume |trig expression| = postive trig exp.

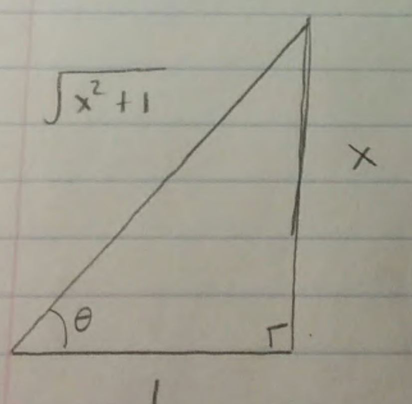
$$= \int \frac{\tan^2\theta}{\sec\theta} \sec^2\theta \, d\theta$$

$$= \int \tan^2\theta \cdot \sec\theta \, d\theta \qquad = \frac{\sec\theta\tan\theta - \ln|\tan\theta + \sec\theta|}{2} + C$$

$$= \int (\sec^2\theta - 1)(\sec\theta) \, d\theta$$

$$= \int \sec^3\theta - \sec\theta \, d\theta$$

$$= \int \sec^3\theta \, d\theta - \int \sec\theta \, d\theta$$



We said that $\tan\theta = x$

at the start.

$$\sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{1}{\frac{1}{\sqrt{x^2+1}}}$$

$$= \sqrt{x^2+1}$$

$$\frac{\sec\theta\tan\theta - \ln|\tan\theta + \sec\theta|}{2} + C$$

$$= \frac{(\sqrt{x^2+1})(x) - \ln|x + \sqrt{x^2+1}|}{2} + C$$

Integral List

1. $\int a\,dx = ax + c$, a is a constant

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$, $n \neq -1$

3. $\int \frac{1}{x}\,dx = \ln|x| + c$

4. $\int e^{kx}\,dx = \dfrac{e^{kx}}{k} + c$

5. $\int a^x\,dx = \dfrac{a^x}{\ln(a)} + c$

6. $\int \cos x\,dx = \sin x + c$

7. $\int \sin x\,dx = -\cos x + c$

8. $\int \sec^2 x\,dx = \tan x + c$

9. $\int \sec x \cdot \tan x\,dx = \sec x + c$

10. $\int \csc^2 x\,dx = -\cot x + c$

11. $\int \csc x \cdot \cot x\,dx = -\csc x + c$

12. $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

13. $\int \csc x\,dx = -\ln|\csc x + \cot x| + c$

14. $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$

15. $\int \dfrac{1}{1+x^2}\,dx = \arctan(x) + c$

16. $\int \dfrac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + c$

17. $\int \ln(x)\,dx = x - x\ln(x) + c$

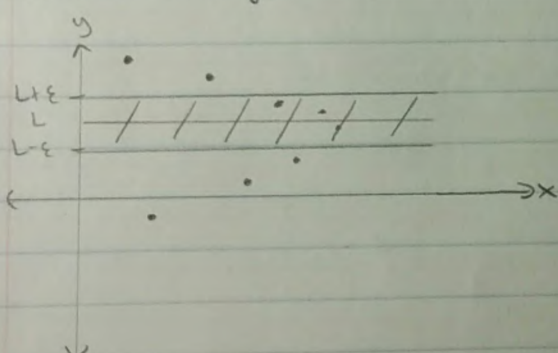
Sequences

- An infinite ordered list of numbers.
- E.g. 1, 2, 3 ...
- A sequence of real numbers is a real-valued function whose domain is $\mathbb{N}$.
- Can be denoted as 1. $\{a_n\}$
  2. $\{a_n\}_{n=1}^{\infty}$
  3. $a_n = f(n)$, where a_n is the general term

Sequences can converge or diverge.

A sequence, $\{a_n\}$, converges to a real number, $L \in \mathbb{R}$ iff $\exists L \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - L| < \varepsilon$

Geometrically:



E.g. Prove that $\left\{ \frac{3}{\sqrt{n} + \sin^2 n} \right\}$ converges to 0

WTS: $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - 0| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary

Choose $N = \frac{9}{\varepsilon^2} > 0$

Note: Choose N at the end of the process.

Suppose $n>N$

Consider $|a_n-0|$

$$= \left| \frac{3}{\sqrt{n}+\sin^2 n} \right|$$

$$= \frac{3}{\sqrt{n}+\sin^2 n}$$ By abs value properties

$$\leq \frac{3}{\sqrt{n}}$$ To minimize the denominator

Since $n>N$, $\frac{1}{n}<\frac{1}{N}$, $\frac{3}{\sqrt{n}}<\frac{3}{\sqrt{N}}$

$$\frac{3}{\sqrt{N}} = \varepsilon$$

$$\sqrt{N} = \frac{3}{\varepsilon}$$

$$N = \frac{9}{\varepsilon^2}$$ This is where you choose N.

$$\frac{3}{\sqrt{N}}$$

$$= \frac{3}{\sqrt{\frac{9}{\varepsilon^2}}}$$

$$= \frac{3}{\left(\frac{3}{\varepsilon}\right)}$$

$= \varepsilon$, as wanted

QED

Sometimes, you need to make a helper function.

E.g. Prove that $a_n = \frac{n^2-2}{n^2+2n+2}$ converges.

If they don't give you L, you have to evaluate the limit of a_n to find it.

$$\lim_{n\to\infty} \frac{n^2-2}{n^2+2n+2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1-\frac{2}{n^2})}{\cancel{n^2}(1+\frac{2}{n}+\frac{2}{n^2})}$$

$$= \lim_{n\to\infty} \frac{1-\cancel{\frac{2}{n^2}}}{1+\cancel{\frac{2}{n}}+\cancel{\frac{2}{n^2}}}$$

$$= 1$$

$\therefore L=1$

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \frac{6}{\varepsilon}$ ______ > 0

Suppose $n > N$

Consider $|a_n - 1|$

$$= \left| \frac{n^2-2}{n^2+2n+2} - 1 \right|$$

$$= \left| \frac{n^2-2-(n^2+2n+2)}{n^2+2n+2} \right|$$

$$= \left| \frac{-2n-4}{n^2+2n+2} \right|$$

$$= \left| \frac{-2(n+2)}{n^2+2n+2} \right|$$

$= \frac{2(n+2)}{n^2+2n+2}$ By abs value properties

$< \frac{2(n+2)}{n^2}$ To min the denominator

$< \frac{2(1+\frac{2}{n})}{n}$ ← If $n=0$, then $(\frac{2}{n})$ would be a problem. We can use a helper function to fix this.

Assume $n > 1$

Since $n > 1$, then $\frac{2}{n} < 2$ and $1 + \frac{2}{n} < 3$

$$\frac{2(1+\frac{2}{n})}{n}$$

$$\leq \frac{6}{n}$$

$$\leq \frac{6}{N} \quad \text{b/c } n > N\text{, so } \frac{1}{n} < \frac{1}{N}$$

$$\frac{6}{N} = \varepsilon$$

$$N = \frac{6}{\varepsilon}$$

However, because of our helper function, $N \geq 1$.
To satisfy both conditions, $N = \max(1, \frac{6}{\varepsilon})$.
Without loss of generality, $N = \frac{6}{\varepsilon}$.

We can ignore the $N \geq 1$ if we say that

$$\frac{6}{N}$$

$$= \frac{6}{(\frac{6}{\varepsilon})}$$

$= \varepsilon$, as wanted

If a sequence diverges, there are 2 ways to prove it.

1. Proof by Contradiction
2. Prove by infinite / negative infinite limit

E.g. of 1

Prove that $\{1+(-1)^n\}$ diverges.

Suppose $\{1+(-1)^n\}$ converges to some number, $L \in \mathbb{R}$.

We know that $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L| < \varepsilon$

Let $\varepsilon = 1$

$\therefore \exists N > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - L| < 1$

When n is odd and if $n > N$:
$$|1-1-L| < 1$$
$$|-L| < 1$$
$$|L| < 1$$
$$-1 < L < 1$$

When n is even and if $n > N$:
$$|1+1-L| < 1$$
$$|2-L| < 1$$
$$|L-2| < 1$$
$$-1 < L-2 < 1$$
$$1 < L < 3$$

L is in both $(-1,1) \cap (1,3)$. But they are disjoint sets, so that's a contradiction.

$\therefore \{1+(-1)^n\}$ diverges

QED

E.g. of 2

Prove $\{n^2\}$ diverges

$\lim_{n \to \infty} n^2 = \infty$

WTS: $\forall M>0, \exists N>0$ s.t. $\forall n \in \mathbb{N}$ if $n>N$ then $a_n > M$

Let $M>0$ be arbitrary.

Choose $N = \underline{\sqrt{M}} > 0$

Suppose $n>N$

Consider n^2

$> N^2$

$N^2 = M$

$N = \sqrt{M}$

N^2

$= (\sqrt{M})^2$

$= M$, as wanted

QED

Theorems

Let $\{a_n\}$ and $\{b_n\}$ be sequences.

If $\underbrace{\lim_{n\to\infty} a_n = a}_{\text{This means } \{a_n\} \text{ converges to } a.}$ and $\underbrace{\lim_{n\to\infty} b_n = b}_{\text{This means } \{b_n\} \text{ converges to } b.}$, for some real numbers a, b.

1. $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$

2. $\lim_{n\to\infty} c(a_n) = c \lim_{n\to\infty} a_n = ca$ Note: 2 is a special case of 3.

3. $\lim_{n\to\infty} (a_n \cdot b_n) = ab$

4. Every converging sequence has a unique limit.

5. Every converging sequence is bounded.
   Note, the converse in general is False.
   I.e. if $\{a_n\}$ is bounded $\not\Rightarrow$ $\{a_n\}$ converges

6. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, $b \neq 0$, $b_n \neq 0$ $\forall n \in \mathbb{N}$

Proof of 1.

Suppose $\{a_n\}$ converges to a ①
$\{b_n\}$ converges to b ②

WTS: $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|(a_n + b_n) - (a+b)| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \underline{\max(N_1, N_2)} > 0$

From ①:
$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \frac{\varepsilon}{2}$

From ②:
$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|b_n - b| < \frac{\varepsilon}{2}$

Suppose $n > N$

Consider $|(a_n + b_n) - (a+b)| < \varepsilon$

$$= |a_n - a + b_n - b| < \varepsilon$$

$$\leq \underbrace{|a_n - a|}_{< \frac{\varepsilon}{2}} + \underbrace{|b_n - b|}_{< \frac{\varepsilon}{2}} \quad \text{(Triangle Inequality)}$$

$< \varepsilon$, as wanted

QED

Proof of 2 and 3

Suppose $\{a_n\}$ converge to a ①

$\{b_n\}$ converge to b ②

WTS: $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n b_n - ab| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary

Choose $N = \underline{\max(N_1, N_2)} > 0$

From ①

$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \frac{\varepsilon}{2b_n}$

From ②

$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|b_n - b| < \frac{\varepsilon}{2a}$

Consider $|a_n b_n - ab|$

$= |a_n b_n - ab_n + ab_n - ab|$

$\leq |a_n - a||b_n| + |b_n - b||a|$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\leq \varepsilon$, as wanted

Proof of 6

Suppose $\{a_n\}$ converges to a. ①

$\{b_n\}$ converges to b. ②

WTS: $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \underline{\max(N_1, N_2)} > 0$

From ①

$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \frac{b\varepsilon}{2}$

From ②

$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|b_n - b| < \frac{(b)(b_n)(\varepsilon)}{2a}$

Suppose $n > N$

Consider $\left| \frac{a_n}{b_n} - \frac{a}{b} \right|$

$$= \left| \frac{a_n(b) - a(b_n)}{(b)(b_n)} \right|$$

$$= \left| \frac{a_n(b) - b_n(a) + (a_n b_n) - (a_n b_n)}{(b)(b_n)} \right|$$

$$= \left| \frac{-a_n(b_n - b) + b_n(a_n - a)}{(b)(b_n)} \right|$$

$$= \left| \frac{b_n(a_n - a)}{b(b_n)} - \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$= \left| \frac{(a_n - a)}{b} - \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$\leq \left| \frac{(a_n - a)}{b} \right| + \left| - \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$\leq \left| \frac{(a_n - a)}{b} \right| + \left| \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\leq \varepsilon$, as wanted

QED

Proof of 4

Suppose that $\{a_n\}$ converges to L_1 and L_2 and $L_1 \neq L_2$

Let $\varepsilon = \frac{|L_1 - L_2|}{2} > 0$ This is to prevent overlapping

$\therefore$

$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L_1| < \frac{|L_1 - L_2|}{2}$

$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L_2| < \frac{|L_1 - L_2|}{2}$

Choose $N = \max(N_1, N_2)$

$$\begin{aligned} |L_1 - L_2| &= |a_n - L_1 + a_n - L_2| \\ &\leq |a_n - L_1| + |a_n - L_2| \quad \text{By Triangle Inequality} \\ &< \left|\frac{L_1 - L_2}{2}\right| + \left|\frac{L_1 - L_2}{2}\right| \\ &< |L_1 - L_2| \end{aligned}$$

$\therefore$ This is a contradiction.

$\therefore$ The limit is unique.

QED

Proof of 5

Suppose $\{a_k\}$ converges to some number, $a \in \mathbb{R}$.

WTS: $\forall \varepsilon > 1$, $\exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \varepsilon$

Let $\varepsilon = 1$

Suppose $n > N$

$|a_n - a| < 1$

By triangle inequality, $|a_n| - |a| < 1$

$|a_n| < |a| + 1$

Let $M = \max\{|a|+1, |a_1|, |a_2| \ldots |a_n|\}$

Then, we have $|a_n| \leq M \; \forall n \in \mathbb{N}$, as wanted

QED

Recursive Sequences

Recursive sequences are sequences such that if there is some index k, so that for $k > k$, the value of a_k is determined by $a_1, a_2, a_3 \dots a_{k-1}$.

E.g.

$a_1 = \sqrt{6}$

$a_{n+1} = \sqrt{6 + a_n}$

In this case, $k = 1$ and for $k > 1$, the value of a_k is determined by $a_1, a_2, \dots a_{k-1}$.

For recursive sequences, use BMCT instead of epsilon proofs.

Bounded Monotone Convergence Theorem (BMCT)

If $\{a_n\}$ is $\underbrace{\text{bounded and monotone,}}$ then $\{a_n\}$ converges.

$\downarrow$

1. Bounded above and strictly increasing

OR

2. Bounded below and strictly decreasing.

Proof of BMCT (Bounded above and strictly increasing.)

Suppose that $\{a_n\}$ is 1. Bounded above
2. Strictly increasing

WTS: $\{a_n\}$ converges $\Leftrightarrow \exists L \in \mathbb{R}, \forall \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - L| < \varepsilon$

Define $A = \{a_n | n \in \mathbb{N}\} \in \mathbb{R}$

Note: $A \neq 0$ because $a_1 \in A$

Furthermore, A is bounded above by statement 1.

By the Completeness Axiom, $\sup(A) = \alpha \in \mathbb{R}$ exists.

Choose $L = \alpha$

Let $\varepsilon > 0$ be arbitrary

Choose $N =$ ______ > 0 Let's pick N's properties instead.

$N \in \mathbb{N}$ s.t. $\alpha - \varepsilon < a_n$

Suppose $n > N$

$\therefore \alpha - \varepsilon < a_N \leq a_n \leq \alpha < \alpha + \varepsilon$

By choice of N

Because $n > N$ and of statement 2

Because $\alpha = \sup\{a_n\}$

because $\varepsilon > 0$

$\Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$

$= -\varepsilon < a_n - \alpha < \varepsilon$

$= |a_n - \alpha| < \varepsilon$

$= |a_n - L| < \varepsilon$, as wanted

QED

Proof of BMCT (Bounded below and strictly decreasing)

Suppose that $\{a_n\}$ is 1. Bounded Below
2. Strictly Decreasing

Define $A = \{a_n \mid n \in \mathbb{N}\} \in \mathbb{R}$
Note: $A \neq 0$ because $a_1 \in A$

Moreover, A is bounded below by statement 1.
$\therefore$ By Completness Axiom, $\inf(A) = \alpha \in \mathbb{R}$ exists

Choose $L = \alpha$
Let $\varepsilon > 0$ be arbitrary
Choose $N = \_\_\_\_ > 0$ Let's pick N's properties instead.
$N \in \mathbb{N}$ s.t. $a_N < \alpha + \varepsilon$.

Suppose $n > N$
So $\alpha - \varepsilon < \alpha \leq a_n \leq a_N < \alpha + \varepsilon$

B/C $\varepsilon > 0$ B/C a_n is bounded below by α. B/c $n > N$ and a_n is bounded below. By definition of N

$\Rightarrow \alpha - \varepsilon < a_N < \alpha + \varepsilon$
$= -\varepsilon < a_n - \alpha < \varepsilon$
$= |a_n - \alpha| < \varepsilon$
$= |a_n - L| < \varepsilon$, as wanted
QED

E.g. Prove that the sequence $a_1 = \sqrt{6}$, $a_{n+1} = \sqrt{6 + a_n}$ if $n > 1$ converges.

1. Have to prove that the sequence is bounded.

Check if it's bounded above or below by rough work first.

$a_1 = \sqrt{6}$
$a_2 = \sqrt{6+\sqrt{6}}$
$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$

They are increasing, so this sequence is bounded above.

Use induction to prove that the sequence is bounded above.

First, use rough work to find which number is bounding the sequence.

$a_1 = \sqrt{6} < \sqrt{9} = 3$ b/c $0 < 6 < 9$
$a_2 = \sqrt{6+\sqrt{6}} < \sqrt{6+3} = 3$
$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}} < \sqrt{6+3} = 3$

$\therefore$ The sequence is bounded by 3.

Induction $\longrightarrow$ ① Base Case:

We know by definition of $\{a_n\}$ that $a_1 = \sqrt{6} < \sqrt{9} = 3$.
This is because $0 < 6 < 9$.

② Induction Hypothesis:

$\forall k \in \mathbb{N}$, if $a_k < 3$ then $a_{k+1} < 3$.
OR $\forall k \in \mathbb{N}$, $a_k < 3 \Rightarrow a_{k+1} < 3$

③ Induction Step:

Let $k \in \mathbb{N}$ be arbitrary.
Suppose $a_k < 3$ (By I.H)

Consider $a_{k+1} = \sqrt{6+\underbrace{a_k}_{<3}}$ (By def of $\{a_n\}$, $k \in \mathbb{N}$) (By I.H)

$< \sqrt{6+3}$

$< \sqrt{9}$

< 3, as wanted

QED

2. Have to prove $\{a_n\}$ is strictly increasing.

3 ways to prove a sequence is increasing.

a) Difference test: $a_{k+1} - a_k \geq 0 \ \forall k \geq 1$

b) Ratio Test: If all terms are positive, and $\frac{a_{k+1}}{a_k} \geq 1 \ \forall k \geq 1$

c) Derivative Test: $a'(x) \geq 0 \ \forall x > 1$, given that $a(x)$ is differentiable on $[1, \infty)$ and $a_k = a(k) \ \forall k \geq 1$.

Note: To prove that a sequence is strictly decreasing, take the opposite of the 3 ways.

Consider $(a_n)^2 - (a_{n+1})^2$

$= (a_n)^2 - (\sqrt{6+a_n})^2$ By def of $\{a_n\}$

$= (a_n)^2 - a_n - 6$

$= (a_n - 3)(a_n + 2)$

Since $a_n < 3$, $a_n - 3 < 0$

$a_n + 2 > 0$

So, $(a_n - 3)(a_n + 2) < 0$

$\therefore \forall n \in \mathbb{N}$, $(a_n)^2 - (a_{n+1})^2 < 0 \iff 0 < (a_n)^2 < (a_{n+1})^2$

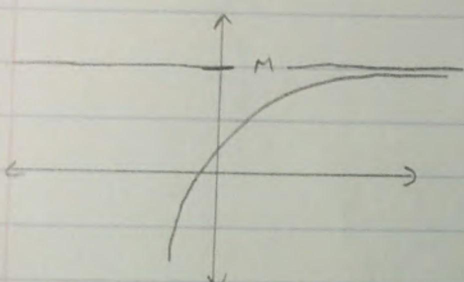
$\therefore a_n < a_{n+1}$, as wanted

QED

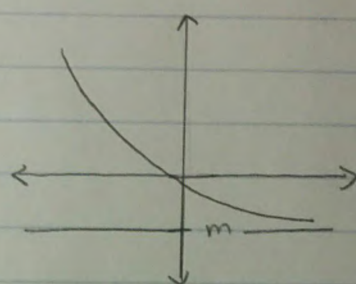
Bounding and Monotone

A sequence $\{a_n\}$ is:

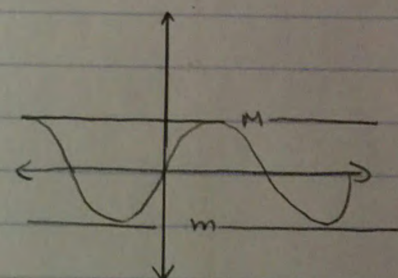
1. Bounded above $\Leftrightarrow \exists M \in \mathbb{R}$ s.t. $a_n \leq M \; \forall n \in \mathbb{N}$
2. Bounded below $\Leftrightarrow \exists m \in \mathbb{R}$ s.t. $a_n \geq m \; \forall n \in \mathbb{N}$
3. Bounded $\Leftrightarrow \{a_n\}$ is bounded above or below.
   $\Leftrightarrow \exists C \in \mathbb{R}, C > 0$ s.t. $|a_n| \leq C$



Bounded above



Bounded Below



Bounded

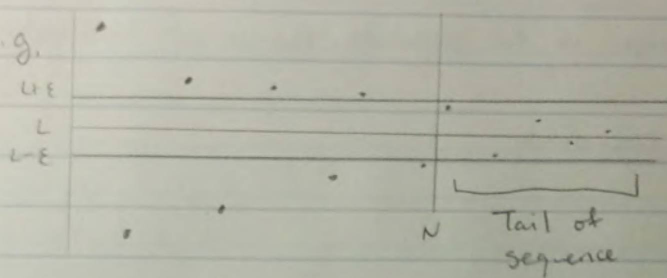
A sequence $\{a_n\}$ is:

1. Increasing $\Leftrightarrow a_n \leq a_{n+1} \; \forall n \in \mathbb{N}$
2. Decreasing $\Leftrightarrow a_n \geq a_{n+1} \; \forall n \in \mathbb{N}$
3. Strictly Increasing $\Leftrightarrow a_n < a_{n+1} \; \forall n \in \mathbb{N}$
4. Strictly Decreasing $\Leftrightarrow a_n > a_{n+1} \; \forall n \in \mathbb{N}$
5. Montone $\Leftrightarrow \{a_n\}$ is either strictly increasing or strictly deereasing

Sequence

- Tail of a sequence
  It is the part of the sequence that is after N.

E.g.



$\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L| < \varepsilon$

Based on this, if the tail of a sequence converges, then the entire sequence converges.

Series

- Denoted as $\sum_{n=1}^{\infty} a_n$, where a_n is the general term.

There are 8 ways to find if a series converges or diverges, but ONLY 2 will give you the sum.

1. By definition
   This will give you the sum of a converging series.
   Only use this if you have a telescoping sum.
   Add the telescoping sum and you will be left with only the first and last term. Then, take the limit of this term as $n \to \infty$. If the series converges, then your limit will be a real number. Otherwise, the series diverges.

2. Geometric Series Test

Will also give you the sum of a converging series.

Suppose $a_n = ar^n$

$\sum_{k=0}^{\infty} ar^n$ will converge if the absolute value of the ratio of

is between 0 and 1, exclusively.

I.e.

Let r = ratio of r^n

If $0<|r|<1$, then the series converges and the sum $= \frac{a}{1-r}$.

Note that a is the first term of the series.

If $|r|>1$, then the series diverges.

Proof

Let $a, r \in \mathbb{R} - \{0\}$

$a + ar + ar^2 + \dots + ar^n = \sum_{n=0}^{\infty} ar^n$

r is the ratio

Let $S_n = a + ar + ar^2 + \dots ar^n$

Let $rS_n = ar + ar^2 + ar^3 + \dots ar^{n+1}$

$S_n - rS_n = a - ar^{n+1}$ (Telescoping sum)

$S_n(1-r) = a(1-r^{n+1})$

Case 1: $r \neq 1$

$$S_n = \frac{a}{1-r}\left(1-r^{n+1}\right)$$

$$= \frac{a}{1-r}\left(\lim_{n\to\infty}\left(1-r^{n+1}\right)\right)$$

$$= \frac{a}{1-r}\left(1-\lim_{n\to\infty} r^{n+1}\right)$$

$$\begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r|>1 \end{cases}$$

Case 2: $r=1$

$S_n = a + a + \dots + a$

$= (n+1)(a)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (n+1)(a)$

$= \text{DNE}$

$\therefore$ Putting it together,

$$\begin{cases} \frac{a}{1-r}, & \text{if } |r|<1 \\ \text{DNE}, & \text{if } |r|\geq 1 \end{cases}$$

3. Divergence Test

Given a series, $\sum a_n$, if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

Note: If $\lim_{n\to\infty} a_n = 0$, this DOES NOT mean the sequence convergences.

Proof: Contrapositive of Vanishing Condition

4. Integral Test

Given a series, $\sum a_n$:

If I have a function, $f(x)$ such that
1. $f(n) = a_n, \forall n \in \mathbb{N}$
2. $f(n)$ is positive, $\forall n \in \mathbb{N}$
3. $f(x)$ is continuous, $\forall n \in \mathbb{N}$
4. $f(x)$ is decreasing on the interval

then, we can use integration to find if our series convergences or not.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \iff \int_1^{\infty} f(x)\,dx \text{ diverges}$$

Note: We don't have to worry about $f(x)$ being continuous, but we do have to prove the other 3.

To prove that $f(x)$ is positive, do a positivity check.

To prove that $f(x)$ is decreasing, find the derivative of $f(x)$ and prove that it's negative.

5. P- Series Test

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}^+$ is called a P- Series.

If $0 < p \leq 1$, then the series diverges.

If $p > 1$, then the series converges.

Proof:

Suppose we have $\sum_{n=1}^{\infty} \frac{1}{n^p}$

We can use the integral test.

Let $f(x) = \frac{1}{x^p}$ on $[1, \infty)$

Positivity check:

$$\frac{1}{x^p} \frac{+}{+} > 0 \text{ on } [1, \infty)$$

Decreasing check:

$$f'(x) = -px^{-p-1}$$

$$= \frac{-p}{x^{p+1}} \frac{-}{+} < 0 \text{ on } [1, \infty)$$

Consider $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x^{-p}\,dx$

$$= \lim_{A \to \infty} \int_1^A x^{-p}\,dx$$

Case 1: $(p=1)$

$$\int_1^{\infty} \frac{1}{x}\,dx$$

$$= \lim_{A \to \infty} \int_1^A \frac{1}{x}\,dx$$

$$= \lim_{A \to \infty} \ln|x| \Big|_1^A$$

$$= \lim_{A \to \infty} (\ln|A| - \underbrace{\ln|1|}_{0})$$

$$= \lim_{A \to \infty} \ln|A|$$

$$= \infty$$

$\therefore$ It diverges

Case 2: $(p \neq 1)$

$$\int_1^{\infty} x^{-p}\,dx$$

$$= \lim_{A \to \infty} \int_1^{A} x^{-p}\,dx$$

$$= \lim_{A \to \infty} \left. \frac{x^{1-p}}{1-p} \right|_1^{A}$$

$$= \frac{1}{1-p} \lim_{A \to \infty} \left(A^{1-p} - 1^{1-p}\right)$$

$$= \frac{1}{1-p} \lim_{A \to \infty} \left(A^{1-p} - 1\right)$$

$\therefore$ If $1-p<0$, or $p>1$, then the series convergences,

Otherwise, it divergences.

QED

6. Comparison Theorem for Series

Note: Try CT if a_n is ugly, has a p-series and/or G.S. in it.

Let Σa_n, Σb_n, Σc_n:

Convergence Case:

If $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$ and b_n convergences, then Σa_n also convergences.

If $0 \leq c_n \leq a_n$ $\forall n \in \mathbb{N}$ and c_n diverges, then Σc_n also divergences.

Proof of Convergence Case:

Suppose 1. $0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$

2. $\sum b_n$ converges

WTS: $\sum a_n$ convergences $\rightarrow \lim_{n \to \infty} S_n$ exists $\rightarrow \{S_n\}$ converges

From 1, $\{S_n\}$ is increasing.

From 2, $\{S_n\}$ is bounded

$\therefore$ By BMCT, $\{S_n\}$ converges.

$\therefore$ $\sum a_n$ convergences

7. Alternating Series / Leibniz Series (AST)

A series in the form of $\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{a_n}_{\Downarrow} = a_1 - a_2 + a_3 \ldots \quad (a_n > 0)$

Positive Part

Given $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, if it satisifies both conditions, then the

series converges,

1. a_n is deecreasing, I.e. $a_n > a_{n+1}$
2. If $\lim_{n \to \infty} a_n = 0$

Note: Try test 2 first. If it fails, use the divergence test. Also, this test can't check for divergence. If either test fails, it means AST is inclusive.

A series absolutely convergences if $\Sigma |a_n|$ convergences.
A series conditionally convergences if $\Sigma |a_n|$ divergences but Σa_n convergences.

8. Ratio Test (RT)

Let Σa_n be a series.

Define $p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Note, $p > 0$

If $p < 1 \Rightarrow \Sigma a_n$ absolutely convergences

If $p > 1 \Rightarrow \Sigma a_n$ divergences

If $p = 1$, the ratio test is inconclusive and use another test

You should use this when you get factorials and complicated products.

Power Series (PS)

Let $\{C_n\} \in \mathbb{R}$, $a \in \mathbb{R}$

Suppose we have $\underbrace{\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 \dots}_{\text{Power Series}}$

C_n: n^{th} term coefficient

$C_n(x-a)^n$: General term

a: Center of Power Series

E.g. $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots x^k$

$C_n = 1$

$a = 0$

Power Series will converge for certain values of x and diverge for others.

Step 1.
Find the radius of convergence using the ratio test.

Step 2.
Check if the end points converge or diverge.

E.g. Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{(n)(4^n)}$

Step 1.
Using the ratio test, consider $\lim_{n\to\infty} \left| \frac{(C_{n+1})(x-a)^{n+1}}{(C_n)(x-a)^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} (x-2)^{n+1}}{(n+1)(4^{n+1})} \div \frac{\cancel{(-1)^n} (x-2)^n}{(n)(4^n)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)\cancel{(x-2)^n}}{(n+1)(4)\cancel{(4^n)}} \cdot \frac{(n)\cancel{(4^n)}}{\cancel{(x-2)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n(x-2)}{4(n+1)} \right|$$

$$= \lim_{n\to\infty} \frac{n|x-2|}{4(n+1)}$$

$$= \frac{|x-2|}{4} \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{|x-2|}{4}$$

By R.T., we know that we get A.C. if $\frac{|x-2|}{4} < 1$

and that we get divergence if $\frac{|x-2|}{4} > 1$.

$$\frac{|x-2|}{4} < 1$$

$= \quad |x-2| < 4$

$\therefore$ The radius is 4.

Step 2.

- Check if $x = a \pm r$ conv or div.

$a = 2$

$r = 4$

1. $x = a - r$

$= 2 - 4$

$= -2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-2-2)^n}{n(4^n)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{(n)(4^n)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n\cancel{(4)}^n}{(n)\cancel{(4)}^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

By P-series, $\frac{1}{n}$ div.

$\therefore$ $x = -2$ div

2. $x = a + r$

$= 2 + 4$

$= 6$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(6-\cancel{2})^n}{(n)\cancel{(4)}^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

We can use AST.

$b_n = \frac{1}{n} > 0$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\left(\frac{1}{n}\right)' = \frac{-1}{n^2} < 0$$

$\therefore$ By AST, $x = 6$ conv

The final answer $= (-2, 6]$.

Properties of Conv Series

Let Σa_n, Σb_n be series.

If $\Sigma a_n = a$ and $\Sigma b_n = b$, for some number a and $b \in \mathbb{R}$

Then:

1. $c \Sigma a_n = \Sigma c a_n = ca$
2. $\Sigma (a_n \pm b_n) = \Sigma a_n \pm \Sigma b_n = a \pm b$
3. $\lim_{n \to \infty} a_n = 0$ (Vanishing Condition)

Note that $\lim_{n \to \infty} a_n = 0$ does not mean a_n convergences.

Proof of Vanishing

Suppose Σa_n conv to a, $a \in \mathbb{R}$

I.e. $\lim_{n \to \infty} S_n = a$, where $S_n = a_1 + a_2 + \dots + a_n$

WTS: $\lim_{n \to \infty} a_n = 0$

$$\lim_{n \to \infty} a_n$$

$$= \lim_{n \to \infty} (a_1 + a_2 + \dots + a_n) - (a_1 + a_2 + \dots + a_{n-1})$$

$$= \lim_{n \to \infty} S_n - S_{n-1}$$

$$= \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1}$$

$$= S - S$$

$= 0$, as wanted

QED